

Twisted waveguide with a Neumann window

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Abstract

This paper is concerned with the study of the existence/non-existence of the discrete spectrum of the Laplace operator on a domain of \mathbb{R}^3 which consists in a twisted tube. This operator is defined by means of mixed boundary conditions. Here we impose Neumann Boundary conditions on a bounded open subset of the boundary of the domain (the Neumann window) and Dirichlet boundary conditions elsewhere.

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1 Introduction

In this work, we would like to study the influence of a geometric twisting on trapped modes which occur in certain waveguides. Here the waveguide consists in a straight tubular domain $\Omega_0 := \mathbb{R} \times \omega$ having a Neumann window on its boundary $\partial\Omega_0$.

The cross section ω is supposed to be an open bounded connected subset of \mathbb{R}^2 of diameter $d > 0$ which is not rotationally invariant. Moreover ω is supposed to have smooth boundary $\partial\omega$.

It can be shown that the Laplace operator associated to such a straight tube has bound states [8].

Let us introduce some notations. Denote by \mathcal{N} the Neumann window. It is an open bounded subset of the boundary $\partial\Omega_0$. Let \mathcal{D} be its complement set in $\partial\Omega_0$. When \mathcal{N} is an annulus of size $l > 0$ we will denote it by,

$$\mathcal{A}_a(l) := I_a(l) \times \partial\omega, I_a(l) := (a, l + a), a \in \mathbb{R}.$$

Consider first the self-adjoint operator $H_0^{\mathcal{N}}$ associated to the following quadratic form. Let $D(Q^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_0) \mid \psi|_{\mathcal{D}} = 0\}$ and for $\psi \in D(Q^{\mathcal{N}})$,

$$Q^{\mathcal{N}}(\psi) = \int_{\Omega_0} |\nabla \psi|^2 dx$$

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i.e. the Laplace operator defined on Ω_0 with Neumann boundary conditions (NBC) on \mathcal{N} and Dirichlet boundary conditions (DBC) on \mathcal{D} [5, 11].

It is actually shown in the Section 2 of this paper that if \mathcal{N} contains an annulus of size l large enough then $H_0^\mathcal{N}$ has at least one discrete eigenvalue. In fact it is proved in [8] that this holds true if \mathcal{N} contains an annulus of any size $l > 0$.

The question we are interested in is the following: is it possible that the discrete spectrum of $H_0^\mathcal{N}$ disappears when we apply a geometric twisting on the guide? This question is motivated by the results of [6, 10] where it is shown that this phenomenon occurs in some bent tubes when they are subjected to a twisting defined from an angle function θ having a derivative $\dot{\theta}$ with a compact support. In this paper we consider the situation described above which is very different from the one of [6, 10].

Let us now define the twisting [4, 7]. Choose $\theta \in C^1_c(\mathbb{R})$ and introduce the diffeomorphism

$$\begin{aligned} \mathcal{L} : \Omega_0 &\longrightarrow \mathbb{R}^3 \\ (s, t_2, t_3) &\longmapsto \left(s, t_2 \cos \theta(s) - t_3 \sin \theta(s), t_2 \sin \theta(s) + t_3 \cos \theta(s) \right). \end{aligned} \quad (1)$$

The twisted tube is given by $\Omega_\theta := \mathcal{L}(\Omega_0)$. Let $D(Q_\theta^\mathcal{N}) = \{\psi \in \mathcal{H}^1(\Omega_\theta) \mid \psi|_{\mathcal{L}(\mathcal{D})} = 0\}$ and consider the following quadratic form

$$Q_\theta^\mathcal{N}(\psi) := \int_{\Omega_\theta} |\nabla \psi|^2 dx, \quad \psi \in D(Q_\theta^\mathcal{N}). \quad (2)$$

Through unitary equivalence, we then have to consider

$$q_\theta^\mathcal{N}(\psi) := Q_\theta^\mathcal{N}(\psi \circ \mathcal{L}^{-1}) = \|\nabla' \psi\|^2 + \|\partial_s \psi + \dot{\theta} \partial_\tau \psi\|^2, \quad (3)$$

$\psi \in D(q_\theta^\mathcal{N}) := \{\psi \in \mathcal{H}^1(\Omega_0) \mid \psi|_{\mathcal{D}} = 0\}$ and where

$$\nabla' := {}^t(\partial_{t_2}, \partial_{t_3}), \quad \partial_\tau := t_2 \partial_{t_3} - t_3 \partial_{t_2}. \quad (4)$$

Denote by $H_\theta^\mathcal{N}$ the associated self-adjoint operator. It is defined as follows (see [5, 11]). Let $D(H_\theta^\mathcal{N}) = \{\psi \in D(q_\theta^\mathcal{N}), \quad H_\theta^\mathcal{N} \psi \in L^2(\Omega_0) \mid \frac{\partial \psi}{\partial n}|_{\mathcal{N}} = 0\}$ with

$$H_\theta^\mathcal{N} \psi = (-\Delta_\omega - (\dot{\theta} \partial_\tau + \partial_s)^2) \psi, \quad (5)$$

where the transverse Laplacian $\Delta_\omega := \partial_{t_2}^2 + \partial_{t_3}^2$. If $\mathcal{N} = \mathcal{A}_a(l), l > 0$, we will denote these forms respectively as Q_θ^l, q_θ^l and the corresponding operator as H_θ^l and if $\mathcal{N} = \emptyset$ we denote the associated operator by H_θ .

Then the main result of this paper is

Theorem 1.1. *i) Under conditions stated above on ω and θ , there exists $l_{\min} := l_{\min}(\omega, d) > 0$ such as if for some $a \in \mathbb{R}$ and $l > l_{\min}$, $\mathcal{N} \supset \mathcal{A}_a(l)$ then*

$$\sigma_d(H_\theta^\mathcal{N}) \neq \emptyset. \quad (6)$$

ii) Suppose in addition that θ has a bounded second derivative. Then there exists $d_{\max} := d_{\max}(\theta, \omega) > 0$ such that for all $0 < d \leq d_{\max}$ there exists $l_{\max} := l_{\max}(\omega, d, \theta)$ such as for all $0 < l \leq l_{\max}$, if $\mathcal{N} \subset \mathcal{A}_a(l)$ and $\text{supp}(\dot{\theta}) \cap I_a(l) = \emptyset$ for some $a \in \mathbb{R}$ then

$$\sigma_d(H_\theta^\mathcal{N}) = \emptyset. \quad (7)$$

Roughly speaking this result implies that for d small enough, the discrete spectrum disappears when the width of the Neumann window decreases.

Let us describe briefly the content of the paper. In the Section 2 we give the proof of the Theorem 1.1 i). The section 3 is devoted to the proof of the second part of the Theorem 1.1, this proof needs several steps. In particular we first establish a local Hardy inequality. This allows us to reduce the problem to the analysis of a one dimensional Schrödinger operator from which the Theorem 1.1 ii) follows. Finally in the Appendix of the paper we give partial results we use in previous sections.

2 Existence of bound states

First we prove the following. Denote by E_1, E_2, \dots the eigenvalues (transverse modes) of the Laplacian $-\Delta_\omega$ defined on $L^2(\omega)$ with DBC on $\partial\omega$. Let χ_1, χ_2, \dots be the associated eigenfunctions. Then we have

Proposition 2.1. $\sigma_{ess}(H_\theta^N) = [E_1, \infty)$.

Proof. We know that $\sigma(H_\theta) = [E_1, \infty)$ see e.g. [2]. But by usual arguments [12], $H_\theta^N \leq H_\theta$, then

$$[E_1, \infty) \subset \sigma_{ess}(H_\theta^N). \quad (8)$$

Let $a' \in \mathbb{R}$ and $l' > 0$ large enough such that $\mathcal{N} \subset \mathcal{A}_{a'}(l') = I_{a'}(l') \times \partial\omega$ and $\text{supp}(\dot{\theta}) \subset I_{a'}(l')$. Let $\tilde{H}_\theta^{l'}$ be the operator defined as in (5) but with additional Neumann boundary conditions on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. So $H_\theta^N \geq \tilde{H}_\theta^{l'}$ and then $\sigma_{ess}(H_\theta^N) \subset \sigma_{ess}(\tilde{H}_\theta^{l'})$ [12].

But $\tilde{H}_\theta^{l'} = \tilde{H}_i \oplus \tilde{H}_e$. The interior operator \tilde{H}_i is the corresponding operator defined on $L^2(I_{a'}(l') \times \omega)$ with NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega \cup (a, a + l) \times \partial\omega$ and DBC elsewhere on the boundary of $\mathcal{A}_{a'}(l')$. By general arguments of [12] it has only discrete spectrum consequently $\sigma_{ess}(\tilde{H}_\theta^{l'}) = \sigma_{ess}(\tilde{H}_e)$.

Now the exterior operator \tilde{H}_e is defined on $L^2((-\infty, a') \times \omega \cup (a' + l', \infty) \times \omega)$ with DBC on $(-\infty, a') \times \partial\omega \cup (a' + l', \infty) \times \partial\omega$ and NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. Since $\theta = 0$ for $x < a'$ and $x > a' + l'$, it is easy to see that

$$\tilde{H}_e = \bigoplus_{n \geq 1} (-\partial^2 + E_n)(\chi_n, \cdot)\chi_n.$$

Hence $\sigma(\tilde{H}_e) = \sigma_{ess}(\tilde{H}_e) = [E_1, +\infty)$. □

The Theorem 1.1 i) follows from

Proposition 2.2. *Under conditions on ω and θ given above, there exists $l_{min} := l_{min}(\omega, d) > 0$ such as for all $l > l_{min}$ we have*

$$\sigma_d(H_\theta^l) \neq \emptyset. \quad (9)$$

Proof. Let $\varphi_{l,a}$ be the following function

$$\varphi_{l,a}(s) := \begin{cases} \frac{10}{l}(s - a), & \text{on } [a, a + \frac{l}{10}); \\ 1, & \text{on } [a + \frac{l}{10}, a + \frac{9l}{10}); \\ -\frac{10}{l}(s - l - a), & \text{on } [a + \frac{9l}{10}, a + l); \\ 0, & \text{elsewhere.} \end{cases}$$

It is easy to see that $\varphi_{l,a} \in D(q_\theta^l)$ and $\|\varphi_{l,a}\|^2 = \frac{13l}{15} |\omega|$. Let us calculate

$$q_\theta^l(\varphi_{l,a}) - E_1 \|\varphi_{l,a}\|^2 = \|\nabla' \varphi_{l,a}\|^2 + \|\dot{\theta} \partial_\tau \varphi_{l,a} + \partial_s \varphi_{l,a}\|^2 - E_1 \|\varphi_{l,a}\|^2. \quad (10)$$

Evidently the first term on the r.h.s of (10) is zero. For the second term on the r.h.s of (10) we get,

$$\|\dot{\theta} \partial_\tau \varphi_{l,a} + \partial_s \varphi_{l,a}\|^2 = \|\partial_s \varphi_{l,a}\|^2 = \frac{20}{l} |\omega|.$$

Then

$$q_\theta^l(\varphi_{l,a}) - E_1 \|\varphi_{l,a}\|^2 = |\omega| \left(\frac{20}{l} - \frac{13l}{15} E_1 \right) \quad (11)$$

and thus if $l \geq l_{min} := \sqrt{\frac{300}{13E_1}}$ we have $q_\theta^l(\varphi_{l,a}) - E_1 \|\varphi_{l,a}\|^2 \leq 0$ \square

2.1 Proof of the Theorem 1.1 i)

Using the same notation as in the Theorem 1.1 i), then $H_\theta^{\mathcal{N}} \leq H_\theta^l$. Moreover these operators have the same essential spectrum, then by the min-max principle the assertion follows.

3 Absence of bound state

In this section we want to prove the second part of the Theorem 1.1. Denote by $\theta_m = \inf(\text{supp}(\dot{\theta}))$, $\theta_M = \sup(\text{supp}(\dot{\theta}))$ and $L = \theta_M - \theta_m$. We first consider the case where the Neumann window is an annulus, $\mathcal{A}_a(l) = I_a(l) \times \omega$.

Proposition 3.1. *Suppose $\mathcal{A}_a(l)$ is such that $a \geq \theta_M$. Assume also that θ has a bounded second derivative. Then there exists $d_{max} := d_{max}(\omega, \theta) > 0$, such that for all $0 < d \leq d_{max}$ there exists $l_{max}(d, \theta, \omega) > 0$ such as for all $0 < l \leq l_{max}$ we have*

$$\sigma_d(H_\theta^l) = \emptyset. \quad (12)$$

Remark 3.2. the case where $l+a \leq \theta_m$ follows from same arguments developed below for the proof of this proposition.

This proof is based on the fact that under conditions of the Proposition 3.1, for every $\psi \in D(q_\theta^l)$ it holds,

$$Q(\psi) := q_\theta^l(\psi) - E_1 \|\psi\|^2 \geq 0. \quad (13)$$

The proof of (13) involves several steps.

3.1 A local Hardy inequality

The aim of this paragraph is to show a Hardy type inequality needed for the proof of the Proposition 3.1. It is the first step of the proof of (13). Let g be the following function

$$g(s) := \begin{cases} 0, & \text{on } I_a(l); \\ E_1, & \text{elsewhere.} \end{cases} \quad (14)$$

Choose $p \in (\theta_m, \theta_M)$ s.t. $\dot{\theta}(p) \neq 0$ and let

$$\rho(s) := \begin{cases} \frac{1}{1+(s-p)^2}, & \text{on } (-\infty, p]; \\ 0, & \text{elsewhere.} \end{cases} \quad (15)$$

Proposition 3.3. *Under same conditions of the Proposition 3.1, then there exists a constant $C > 0$ depending on p and ω and $\dot{\theta}$ such that for any $\psi \in D(q_\theta^l)$,*

$$\|\nabla' \psi\|^2 + \|\dot{\theta} \partial_\tau \psi + \partial_s \psi\|^2 - \int_{\Omega_0} g(s) |\psi|^2 ds dt \geq C \int_{\Omega_0} \rho(s) |\psi|^2 ds dt. \quad (16)$$

We first show the following lemma. Denote by $\Omega_p := (-\infty, p) \times \omega$.

Lemma 3.4. *Under same conditions of the Proposition 3.3. Then for any $\psi \in D(q_\theta^l)$ we have*

$$\int_{\Omega_p} |\nabla' \psi|^2 + |\dot{\theta} \partial_\tau \psi + \partial_s \psi|^2 - E_1 |\psi|^2 ds dt \geq C \int_{\Omega_p} \rho(s) |\psi|^2 ds dt. \quad (17)$$

In the following we will use notations suggested in [6]. For $A \subset \mathbb{R}$ denote by χ_A the characteristic function of $A \times \omega$. Let $\psi \in D(q_\theta^l)$ and define,

$$\begin{aligned} q_1^A(\psi) &:= \|\chi_A \nabla' \psi\|^2 - E_1 \|\chi_A \psi\|^2, & q_2^A(\psi) &:= \|\chi_A \partial_s \psi\|^2, \\ q_3^A(\psi) &:= \|\chi_A \dot{\theta} \partial_\tau \psi\|^2, & q_{2,3}^A(\psi) &:= 2\Re(\partial_s \psi, \chi_A \dot{\theta} \partial_\tau \psi). \end{aligned} \quad (18)$$

Denote also by $Q^A(\psi) = q_1^A(\psi) + q_2^A(\psi) + q_3^A(\psi) + q_{2,3}^A(\psi)$. Here and hereafter we often use the fact that for any $\psi \in D(q_\theta^l)$

$$q_1^A(\psi) \geq 0, \quad (19)$$

for every $A \subset \mathbb{R}$ such that $A \cap I_a(l) = \emptyset$.

Proof. Choose $r > 0$ such that $\dot{\theta}(s) \neq 0$ for any $s \in [p-r, p]$. Let f be the following function:

$$f(s) := \begin{cases} 0, & \text{on } (p, \infty); \\ \frac{p-s}{r}, & \text{on } (p-r, p]; \\ 1, & \text{elsewhere.} \end{cases} \quad (20)$$

For any $\psi \in D(q_\theta^l)$, simple estimates lead to:

$$\begin{aligned} \int_{\Omega_p} \frac{|\psi(s, t)|^2}{1+(s-p)^2} ds dt &= \int_{\Omega_p} \frac{|\psi(s, t)f(s) + (1-f(s))\psi(s, t)|^2}{1+(s-p)^2} ds dt \\ &\leq 2 \left(\int_{\Omega_p} \frac{|f(s)\psi(s, t)|^2}{(s-p)^2} ds dt + \|\chi_{(p-r, p)} \psi\|^2 \right). \end{aligned} \quad (21)$$

Since $f(p)\psi(p, \cdot) = 0$, we can use the usual Hardy inequality (see e.g. [9]), then we get,

$$\int_{\Omega_p} \frac{|\psi(s, t)|^2}{1+(s-p)^2} ds dt \leq 8q_2^{(-\infty, p)}(f\psi) + 2\|\chi_{(p-r, p)} \psi\|^2. \quad (22)$$

Note that with our choice $[p-r, p] \cap [a, a+l] = \emptyset$. Hence to estimate the second term on the r.h.s of (22) we use the Theorem 6.5 of [10], then there exists $\lambda_0 = \lambda_0(\theta, p, r) > 0$ s.t. for any $\psi \in D(q_\theta^l)$ we have

$$\|\chi_{(p-r, p)}\psi\|^2 \leq \frac{1}{\lambda_0} Q^{(p-r, p)}(\psi) \leq \frac{1}{\lambda_0} Q^{(-\infty, p)}(\psi). \quad (23)$$

We now want to estimate the first term on the right hand side of (22). We have

$$q_2^{(-\infty, p)}(f\psi) = \int_{\Omega_p} |\partial_s(f\psi)|^2 ds dt = q_2^{(-\infty, \theta_m)}(f\psi) + q_2^{(\theta_m, p)}(f\psi). \quad (24)$$

Evidently since $\dot{\theta} = 0$ and $f = 1$ in $(-\infty, \theta_m)$, from (19), we have

$$q_2^{(-\infty, \theta_m)}(f\psi) \leq Q^{(-\infty, \theta_m)}(\psi). \quad (25)$$

In the other hand since $f(p)\psi(p, \cdot) = 0$, we can apply the Lemma 4.1 of the Appendix. So for any $0 < \alpha < 1$ there exists $\gamma_{\alpha, 1} > 0$ such that

$$|q_{2,3}^{(\theta_m, p)}(f\psi)| \leq \gamma_{\alpha, 1} q_1^{(\theta_m, p)}(f\psi) + \alpha q_2^{(\theta_m, p)}(f\psi) + q_3^{(\theta_m, p)}(f\psi). \quad (26)$$

Let $\gamma := \max(1, \gamma_{\alpha, 1})$. Then

$$\gamma^{-1} |q_{2,3}^{(\theta_m, p)}(f\psi)| \leq q_1^{(\theta_m, p)}(f\psi) + \alpha \gamma^{-1} q_2^{(\theta_m, p)}(f\psi) + \gamma^{-1} q_3^{(\theta_m, p)}(f\psi). \quad (27)$$

Hence with the decomposition, $q_{2,3}^{(\theta_m, p)} = \gamma^{-1} q_{2,3}^{(\theta_m, p)} + (1 - \gamma^{-1}) q_{2,3}^{(\theta_m, p)}$ and (27) we have,

$$\begin{aligned} Q^{(\theta_m, p)}(f\psi) &\geq (1 - \gamma^{-1}) \left(q_2^{(\theta_m, p)}(f\psi) + q_{2,3}^{(\theta_m, p)}(f\psi) + q_3^{(\theta_m, p)}(f\psi) \right) \\ &\quad + \gamma^{-1} (1 - \alpha) q_2^{(\theta_m, p)}(f\psi) \end{aligned} \quad (28)$$

and since $q_3^{(\theta_m, p)} + q_{2,3}^{(\theta_m, p)} + q_2^{(\theta_m, p)} \geq 0$, we arrive at,

$$q_2^{(\theta_m, p)}(f\psi) \leq \frac{\gamma}{(1 - \alpha)} Q^{(\theta_m, p)}(f\psi). \quad (29)$$

Now by using that, $q_1^{(\theta_m, p)}(f\psi) \leq q_1^{(\theta_m, p)}(\psi)$,

$$\|\chi_{(\theta_m, p)}(\partial_s + \dot{\theta} \partial_\tau)(f\psi)\|^2 \leq 2(\|\chi_{(\theta_m, p)}(\partial_s + \dot{\theta} \partial_\tau)\psi\|^2 + \frac{1}{r^2} \|\chi_{(p-r, p)}\psi\|^2)$$

and (23), we get,

$$q_2^{(\theta_m, p)}(f\psi) \leq \frac{2\gamma}{(1 - \alpha)} (Q^{(\theta_m, p)}(\psi) + \frac{1}{\lambda_0 r^2} Q^{(p-r, p)}(\psi)) \leq c' Q^{(\theta_m, p)}(\psi) \quad (30)$$

with $c' = \frac{2\gamma}{(1 - \alpha)} (1 + \frac{1}{\lambda_0 r^2})$. Then (25) and (30) imply

$$q_2^{(-\infty, p)}(f\psi) \leq (1 + c') Q^{(-\infty, p)}(\psi). \quad (31)$$

Hence (31) and (23) prove the lemma with

$$C^{-1} = 8(1 + c') + \frac{2}{\lambda_0}. \quad (32)$$

□

Proof of the proposition 3.3. To prove the proposition we note that for any $\psi \in D(q_\theta^l)$ and for $p' \in \mathbb{R}$ we have

$$\int_{\omega} \int_{p'}^{\infty} |\nabla' \psi|^2 + |\dot{\theta} \partial_\tau \psi + \partial_s \psi|^2 ds dt \geq \int_{\omega} \int_{p'}^{\infty} g(s) |\psi|^2 ds dt. \quad (33)$$

Then (33) with $p' = p$ and Lemma 3.4 imply (16). □

3.2 Reduction to a one dimensional problem

We now want to prove the following result,

Proposition 3.5. *Under conditions of the Proposition 3.1, then a sufficient condition in order to get (13) is given by*

$$\int_{\mathbb{R}} |\psi'(s)|^2 + 2C\rho(s) |\psi(s)|^2 ds - 4E_1 \int_a^{a+l} |\psi(s)|^2 ds \geq 0, \quad (34)$$

for any $\psi \in \mathcal{H}^1(\mathbb{R})$ where the constant C is defined in (32).

Remark 3.6. This proposition means that the positivity needed here is given by the positivity of the effective one dimensional Schrödinger operator on $L^2(\mathbb{R})$,

$$-\frac{d^2}{ds^2} + 2C\rho(s) - 4E_1 \mathbf{1}_{I_a(l)}. \quad (35)$$

where $\mathbf{1}_{I_a(l)}$ is the characteristic function of $I_a(l)$.

Proof. Evidently we have

$$Q(\psi) = \frac{1}{2} \left(Q(\psi) - \int_{\Omega_0} (E_1 - g(s)) |\psi|^2 ds dt + q_\theta^l(\psi) - \int_{\Omega_0} g(s) |\psi|^2 ds dt \right), \quad (36)$$

where g is defined in (14). By using (16), then

$$Q(\psi) \geq \frac{1}{2} \left(q_\theta^l(\psi) - E_1 \|\psi\|^2 + C \int_{\Omega_0} \rho(s) |\psi|^2 ds dt - E_1 \|\chi_{(a, a+l)} \psi\|^2 \right) \quad (37)$$

Rewrite the expression of q_θ^l given by (3) as follows:

$$q_\theta^l(\psi) = \|\nabla' \psi\|^2 + \|\partial_s \psi\|^2 + \|\dot{\theta} \partial_\tau \psi\|^2 + 2\Re(\partial_s \psi, \dot{\theta} \partial_\tau \psi). \quad (38)$$

We estimate the last term of the r.h.s. of (38). By using of the formula (49) of the Appendix,

$$|q_{2,3}(\psi)| = |q_{2,3}^{(\theta_m, \theta_M)}(\psi)| \leq \gamma_{\frac{1}{2}, \frac{1}{2}} q_1^{(\theta_m, \theta_M)}(\psi) + \frac{1}{2} q_2^{(\theta_m, \theta_M)}(\psi) + \frac{1}{2} q_3^{(\theta_m, \theta_M)}(\psi) \quad (39)$$

where

$$\gamma_{\frac{1}{2}, \frac{1}{2}} := \tilde{\gamma}_{\frac{1}{2}, \frac{1}{2}} + 4d^2 \|\dot{\theta}\|_\infty^2 \quad (40)$$

with $\tilde{\gamma}_{\frac{1}{2}, \frac{1}{2}} := \max\left\{\frac{d\|\dot{\theta}\|_\infty\|\ddot{\theta}\|_\infty\sqrt{f(L)}}{\theta_0\sqrt{\lambda}}, \frac{d^2\|\ddot{\theta}\|_\infty^2 f(L)}{\lambda\theta_0^2}, 2d^2\|\ddot{\theta}\|_\infty^2 f(L)\right\}$ for some constant $\lambda > 0$ depending only on the section ω and $f(L) := \max\{2 + \frac{16L^2}{r^2}, 4L^2\}$.

Hence (38) together with (39) give:

$$q_\theta^l(\psi) \geq \|\nabla' \psi\|^2 + \frac{1}{2} \|\partial_s \psi\|^2 + \frac{1}{2} \|\dot{\theta} \partial_\tau \psi\|^2 - \gamma_{\frac{1}{2}, \frac{1}{2}} q_1^{(\theta_m, \theta_M)}(\psi). \quad (41)$$

In view of (19) we have

$$\|\nabla' \psi\|^2 - E_1 \|\psi\|^2 \geq q_1^{(\theta_m, \theta_M)}(\psi) + q_1^{I_a(l)}(\psi) \geq q_1^{(\theta_m, \theta_M)}(\psi) - E_1 \|\chi_{(a, a+l)} \psi\|^2.$$

Thus this last inequality together with (41) in (37) give

$$\begin{aligned} Q(\psi) &\geq \frac{1}{2} \left(\frac{1}{2} \|\partial_s \psi\|^2 + \frac{1}{2} \|\dot{\theta} \partial_\tau \psi\|^2 + C \int_{\Omega_0} \rho(s) |\psi|^2 ds dt - 2E_1 \|\chi_{(a, a+l)} \psi\|^2 \right. \\ &\quad \left. + (1 - \gamma_{\frac{1}{2}, \frac{1}{2}}) q_1^{(\theta_m, \theta_M)}(\psi) \right). \end{aligned}$$

Now if $0 < d \leq d_{max}$ then $\gamma_{\frac{1}{2}, \frac{1}{2}} \leq 1$ so the Proposition 3.5 follows.

3.3 The one dimensional Schrödinger operator

In this part, under our conditions, we want to show that the one dimensional Schrödinger operator (35) is a positive operator. In view of the Proposition 3.5 this will imply the Proposition 3.1. Here we follow a similar strategy as in [1].

Proposition 3.7. *for all $\varphi \in \mathcal{H}^1(\mathbb{R})$, then there exists $l_{max} > 0$ such that for any $0 < l \leq l_{max}$ we have*

$$\int_{\mathbb{R}} |\varphi'(s)|^2 + 2C\rho(s) |\varphi(s)|^2 ds \geq 4E_1 \int_{I_a(l)} |\varphi(s)|^2 ds. \quad (42)$$

Proof. Introduce the following function:

$$\Phi(s) := \begin{cases} (\frac{\pi}{2} + \arctan(s - p)), & \text{if } s < p; \\ \frac{\pi}{2}, & \text{if } s \geq p. \end{cases} \quad (43)$$

where p is the same real number as in (15). So clearly $\Phi' = \rho$. For any $t \in I_a(l)$ and $\varphi \in \mathcal{H}^1(\mathbb{R})$, we have:

$$\begin{aligned} \frac{\pi}{2} \varphi(t) = \Phi(t) \varphi(t) &= \int_{-\infty}^t (\Phi(s) \varphi(s))' ds \\ &= \int_{-\infty}^t \rho(s) \varphi(s) ds + \int_{-\infty}^t \Phi(s) \varphi'(s) ds \end{aligned} \quad (44)$$

and since $\rho(s) = 0$ for any $s \in (p, \infty)$, we get,

$$\frac{\pi}{2}\varphi(t) = \int_{-\infty}^p \rho(s)\varphi(s)ds + \int_{-\infty}^t \Phi(s)\varphi'(s)ds. \quad (45)$$

Then some straightforward estimates lead to,

$$\begin{aligned} \frac{\pi^2}{4}\varphi^2(t) &\leq 2\left(\left(\int_{-\infty}^p \rho(s)\varphi(s)ds\right)^2 + \left(\int_{-\infty}^t \Phi(s)\varphi'(s)ds\right)^2\right) \\ &\leq 2\left(\int_{-\infty}^p \rho(s)ds \int_{-\infty}^p \rho(s)\varphi^2(s)ds + \int_{-\infty}^t \Phi^2(s)ds \int_{-\infty}^t \varphi'^2(s)ds\right). \end{aligned} \quad (46)$$

By direct calculation $\int_{-\infty}^p \rho(s)ds = \frac{\pi}{2}$ and $\int_{-\infty}^p \Phi^2(s)ds + \int_p^t \Phi^2(s)ds = \pi \ln 2 + \frac{\pi^2}{4}(t-p)$. Hence we get,

$$|\varphi(t)|^2 \leq \frac{4}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \left(\frac{8\ln 2}{\pi} + 2(t-p)\right) \int_{\mathbb{R}} |\varphi'(s)|^2 ds \quad (47)$$

We integrate both sides of (47) over $I_a(l)$, then

$$\begin{aligned} \int_{I_a(l)} |\varphi(t)|^2 dt &\leq \frac{4l}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \left(\left(\frac{8\ln 2}{\pi} + 2(a-p)\right)l + l^2\right) \int_{\mathbb{R}} |\varphi'(s)|^2 ds \\ &\leq c'' \int_{\mathbb{R}} 2C\rho(s)\varphi^2(s) + |\varphi'(s)|^2 ds \end{aligned}$$

where $c'' = 2l\left(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a-p\right) + l^2$. Finally we get,

$$4E_1 \int_a^{l+a} |\varphi(t)|^2 dt \leq 4E_1 c'' \int_{\mathbb{R}} 2C\rho(s) |\varphi(s)|^2 + |\varphi'(s)|^2 ds. \quad (48)$$

So choose $0 < l \leq l_{max}$ with

$$l_{max} := -\left(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a-p\right) + \sqrt{\left(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a-p\right)^2 + (4E_1)^{-1}}$$

then $4E_1 c'' \leq 1$ and the proposition 3.7 follows. \square

3.4 proof of the Theorem 1.1 ii)

Under assumptions of the Theorem 1.1 ii) $H_{\theta}^{\mathcal{N}} \geq H_{\theta}^l$. These two operators have the same essential spectrum so the Theorem 1.1 ii) is proved by applying the Proposition 3.1 and the min-max principle.

4 Appendix

In this appendix we give a slight extension of the lemma 3 of [6] which states that under our conditions, for all $\psi \in D(q_{\theta}^l)$ we have for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha, \beta} > 0$ such that:

$$|q_{2,3}(\psi)| \leq \gamma_{\alpha, \beta} q_1(\psi) + \alpha q_2(\psi) + \beta q_3(\psi). \quad (49)$$

Then we have

Lemma 4.1. *Let $p \in (\theta_m, \theta_M)$. For all $\psi \in D(q_\theta^l)$ such that $\psi(p, \cdot) = 0$, then for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha, \beta} > 0$ such that:*

$$|q_{2,3}^{(\theta_m, p)}(\psi)| \leq \gamma_{\alpha, \beta} q_1^{(\theta_m, p)}(\psi) + \alpha q_2^{(\theta_m, p)}(\psi) + \beta q_3^{(\theta_m, p)}(\psi). \quad (50)$$

Proof. Let $\psi \in D(q_\theta^l)$ such that $\psi(p, \cdot) = 0$. Then $\psi \in \mathcal{H}_0^1(\Omega_p)$. We know that we may first consider vectors $\psi(s, t) = \chi_1(t)\phi(s, t)$, where $\phi \in C_0^\infty(\Omega_p)$. For such a vector ψ we have,

$$\begin{aligned} q_1^{(\theta_m, p)}(\psi) &= \|\chi_{(\theta_m, p)} \chi_1 \nabla' \phi\|^2, & q_2^{(\theta_m, p)}(\psi) &= \|\chi_{(\theta_m, p)} \chi_1 \partial_s \phi\|^2 \\ q_3^{(\theta_m, p)}(\psi) &= \|\chi_{(\theta_m, p)} \dot{\theta}(\chi_1 \partial_\tau \phi + \phi \partial_\tau \chi_1)\|^2 \end{aligned} \quad (51)$$

and

$$q_{2,3}^{(\theta_m, p)}(\psi) = 2(\dot{\theta} \chi_{(\theta_m, p)} \chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi) + 2(\dot{\theta} \chi_{(\theta_m, p)} \phi \partial_\tau \chi_1, \chi_1 \partial_s \phi) \quad (52)$$

By using simple estimates the first term on the r.h.s of (52) is estimated as :

$$|2(\dot{\theta} \chi_{(\theta_m, p)} \chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi)| \leq 2 \|\dot{\theta}\|_\infty \|\chi_{(\theta_m, p)} \chi_1 \nabla' \phi\| \|\chi_{(\theta_m, p)} \chi_1 \partial_s \phi\|$$

then

$$|2(\dot{\theta} \chi_{(\theta_m, p)} \chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi)| \leq c_1 q_1^{(\theta_m, p)}(\psi) + \frac{\alpha}{2} q_2^{(\theta_m, p)}(\psi), \quad (53)$$

where $c_1 := \frac{2}{\alpha} d^2 \|\dot{\theta}\|_\infty^2$ and $\alpha > 0$.

Integrating by parts twice and using the fact that $\dot{\theta}(\theta_m) = \phi(p, \cdot) = 0$, the second term of the r.h.s of (52) is written as

$$2(\dot{\theta} \chi_{(\theta_m, p)} \phi \partial_\tau \chi_1, \chi_1 \partial_s \phi) = (\chi_{(\theta_m, p)} \ddot{\theta} \phi \chi_1, \chi_1 \partial_\tau \phi). \quad (54)$$

Then the Cauchy Schwartz inequality implies,

$$|(\chi_{(\theta_m, p)} \ddot{\theta} \phi \chi_1, \chi_1 \partial_\tau \phi)|^2 \leq d^2 \|\ddot{\theta}\|_\infty^2 q_1^{(\theta_m, p)} \|\chi_{(\theta_m, p)} \chi_1 \phi\|^2. \quad (55)$$

Let $p' \in \mathbb{R}$ and $r' > 0$ such that $(p' - r, p') \subset (\theta_m, p)$ and for $s \in (p' - r, p')$, $|\dot{\theta}(s)| \geq \dot{\theta}_0$ for some $\dot{\theta}_0 > 0$. As in the proof of the Lemma 3 of [6] we have,

$$\|\chi_{(\theta_m, p)} \chi_1 \phi\|^2 \leq c_2 \left(q_2^{(\theta_m, p)}(\psi) + \dot{\theta}_0^{-2} \|\chi_{(p'-r, p')} \dot{\theta} \chi_1 \phi\|^2 \right) \quad (56)$$

where $c_2 := \max \left\{ 2 + 16 \frac{(p-\theta_m)^2}{r^2}, 4(p-\theta_m)^2 \right\}$.

Moreover, for any $s \in \mathbb{R}$, $\dot{\theta}(s) \chi_1 \phi(s, \cdot) \in \mathcal{H}_0^1(\Omega_p)$, then by using the Lemma 1 of [6] there exists $\lambda > 0$ depending on ω such that :

$$\|\chi_{(p'-r, p')} \dot{\theta} \chi_1 \phi\|^2 \leq \|\chi_{(\theta_m, p)} \dot{\theta} \chi_1 \phi\|^2 \leq \lambda^{-1} \left(q_3^{(\theta_m, p)}(\psi) + \|\dot{\theta}\|_\infty^2 q_1^{(\theta_m, p)}(\psi) \right). \quad (57)$$

Hence (56), (57) and (54) give

$$|(\chi_{(\theta_m, p)} \ddot{\theta} \phi \chi_1, \chi_1 \partial_\tau \phi)|^2 \leq \left(c_3 q_1^{(\theta_m, p)}(\psi) + \frac{\alpha}{2} q_2^{(\theta_m, p)}(\psi) + \beta q_3^{(\theta_m, p)}(\psi) \right)^2 \quad (58)$$

where $c_3 := \max \left\{ \frac{d \|\ddot{\theta}\|_\infty \sqrt{c_2}}{\dot{\theta}_0 \sqrt{\lambda}}, \frac{d^2 \|\ddot{\theta}\|_\infty^2 c_2}{\alpha}, \frac{d^2 \|\ddot{\theta}\|_\infty^2 c_2}{2\beta \dot{\theta}_0^2 \lambda} \right\}$. Then (53) and (58) imply (50) with $\gamma_{\alpha, \beta} := c_1 + c_3$.

Note that we can choose $\chi_1 > 0$ on ω . So that (50) holds for every $\psi \in C_0^\infty(\Omega_p)$ and by a density argument this is even true for $\psi \in \mathcal{H}_0^1(\Omega_p)$.

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